

ON k -CAPS IN $\mathbf{PG}(n, q)$, WITH q EVEN AND $n \geq 3$

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ABSTRACT. Let $m_2(n, q)$ be the maximum size of k for which there exists a k -cap in $\mathbf{PG}(n, q)$, and let $m'_2(n, q)$ be the second largest value of k for which there exists a complete k -cap in $\mathbf{PG}(n, q)$. In this paper Chao's upper bound $q^2 - q + 5$ for $m'_2(3, q)$, q even and $q \geq 8$, will be improved. As a corollary new bounds for $m_2(n, q)$, q even, $q \geq 8$ and $n \geq 4$, are obtained. Cao and Ou published a better bound but there seems to be a gap in their proof.

Keywords: projective space, finite field, k -cap

1. INTRODUCTION

A k -arc of $\mathbf{PG}(2, q)$ is a set of k points, no three of which are collinear; a k -cap of $\mathbf{PG}(n, q)$, $n \geq 3$, is a set of k points, no three of which are collinear. A k -arc or k -cap is *complete* if it is not contained in a $(k+1)$ -arc or $(k+1)$ -cap. The largest value of k for which a k -arc of $\mathbf{PG}(2, q)$, or a k -cap of $\mathbf{PG}(n, q)$ with $n \geq 3$, exists is denoted by $m_2(n, q)$. The size of the second largest complete k -arc of $\mathbf{PG}(2, q)$ or k -cap of $\mathbf{PG}(n, q)$, $n \geq 3$, is denoted by $m'_2(n, q)$.

Theorem 1.1. (i) $m_2(2, q) = q + 2$, q even [7];
(ii) $m_2(3, q) = q^2 + 1$, q even, $q > 2$ [6, 1, 10];
(iii) $m_2(n, 2) = 2^n$ [1];
(iv) $m_2(4, 4) = 41$ [5];
(v) $m'_2(n, 2) = 2^{n-1} + 2^{n-3}$ [4];
(vi) $m'_2(3, 4) = 14$ [8].

Theorem 1.2 ([11, 13, 7]). Let K be a k -arc of $\mathbf{PG}(2, q)$, q even and $q > 2$, with $q - \sqrt{q} + 1 < k \leq q + 1$. Then K can be uniquely extended to a $(q+2)$ -arc of $\mathbf{PG}(2, q)$.

For any k -arc K in $\mathbf{PG}(2, q)$ or k -cap K in $\mathbf{PG}(n, q)$, $n \geq 3$, a *tangent* of K is a line which has exactly one point in common with K . Let t be the number of tangents of K through a point P of K and let $\sigma_1(Q)$ be the number of tangents of K through a point $Q \notin K$. Then for a k -arc K $t + k = q + 2$ and for a k -cap K $t + k = q^{n-1} + q^{n-2} + \cdots + q + 2$.

Theorem 1.3 ([8]). If K is a complete k -arc in $\mathbf{PG}(2, q)$, q even, or a complete k -cap in $\mathbf{PG}(n, q)$, $n \geq 3$ and q even, then $\sigma_1(Q) \leq t$ for each point Q not on K .

Theorem 1.4 ([3]).

$$(1) \quad m'_2(3, q) \leq q^2 - q + 5, q \text{ even}, q \geq 8.$$

To prove Theorem 1.4 J.-M. Chao relies on the following crucial lemma.

Lemma 1.5 ([3]). *Let K be a complete k -cap in $\mathbf{PG}(3, q)$ with q even. If Π is a plane such that $|\Pi \cap K| = x$, then*

$$(2) \quad t(t-1) \geq q(q+2-x)x.$$

In the underlying paper the following improvement of Chao's result will be obtained.

Theorem 1.6 (Main Theorem).

$$(3) \quad m'_2(3, q) < q^2 - (\sqrt{5} - 1)q + 5, q \text{ even}, q \geq 8.$$

As a corollary new bounds for $m_2(n, q)$, q even, $q \geq 8$ and $n \geq 4$, are obtained.

Combining the main theorem of [12] with Theorem 1.6, there is an immediate improvement of the upper bound for $m'_2(3, q)$, $q \geq 2048$. We thank T. Szőnyi for bringing reference [12] to our attention.

Theorem 1.7.

$$(4) \quad m'_2(3, q) < q^2 - 2q + 3\sqrt{q} + 2, q \text{ even}, q \geq 2048.$$

2. A FIRST IMPROVEMENT OF CHAO'S BOUND

Theorem 2.1.

$$(5) \quad m'_2(3, q) \leq q^2 - q + 3, q \text{ even}, q \geq 8.$$

Proof. Let K be a complete k -cap in $\mathbf{PG}(3, q)$, q even, $q \geq 8$ and $k < q^2 + 1$. Let Π be a plane of $\mathbf{PG}(3, q)$ for which $4 \leq |\Pi \cap K| \leq q - 2$. Let $f(X) = q(q + 2 - X)X$. Then

$$(6) \quad t(t-1) \geq f(4) = f(q-2) = 4q(q-2),$$

by Lemma 1.5. So

$$(7) \quad t \geq \frac{1 + \sqrt{1 + 16q(q-2)}}{2} \geq 2q - \frac{7}{4} \text{ for } q \geq 8.$$

Hence $k \leq q^2 + q + 2 - 2q + \frac{7}{4} = q^2 - q + \frac{15}{4}$, and consequently $k \leq q^2 - q + 3$.

So we may assume that either $|\Pi \cap K| \leq 3$, or $|\Pi \cap K| \geq q - 1$, for any plane Π of $\mathbf{PG}(3, q)$. Let l_1, l_2, \dots, l_t be the t tangents of K through the point $P \in K$. We consider three cases depending on the number of planes containing l_i and intersecting K in at most 3 points.

(A) There exists exactly one plane Π_{l_i} containing l_i such that $|\Pi_{l_i} \cap K| \leq 3$, $i = 1, 2, \dots, t$. We will show that in this case $k \leq q^2 - q + 3$.

Assume there is exactly one plane Π through P with $|\Pi \cap K| \leq 3$. Then for $i = 1, 2, \dots, t$, $\Pi_{l_i} = \Pi$. Hence all tangents of K containing P are in Π . So $t \leq q + 1$,

a contradiction. Hence there are at least two planes Π_1, Π_2 through P such that $|\Pi_i \cap K| \leq 3, i = 1, 2$. Then $|\Pi_1 \cap \Pi_2 \cap K| = 2$. Consequently $t \geq 2(q-1)$, and so $k \leq q^2 + q + 2 - 2q + 2 = q^2 - q + 4$.

Assume, by way of contradiction, that $k = q^2 - q + 4$. So $t = 2(q-1)$. Then $|\Pi_1 \cap K| = |\Pi_2 \cap K| = 3$. All tangent lines at P are contained in Π_1 and Π_2 . Let l be a tangent of K at P in Π_1 , and consider the $q+1$ planes containing l . The plane Π_1 is the only of these planes which intersects K in 3 points, exactly $q-1$ planes through l contain 2 tangent lines at P and so intersect K in a q -arc and the remaining plane through l contains exactly one tangent line at P and so intersects K in a $(q+1)$ -arc.

Let $\tilde{\Pi}$ be the unique plane containing l which intersects K in a $(q+1)$ -arc, let $\tilde{\Pi} \cap K = O$, and let N be the kernel of O , that is, N is the unique point of $\tilde{\Pi}$ which extends O to a $(q+2)$ -arc of $\tilde{\Pi}$. Clearly $N \in l$.

If K' is a k' -arc of a plane $\mathbf{PG}(2, q)$ and $P' \in \mathbf{PG}(2, q) \setminus K'$, then the parity of the number of tangents of K' through P' is the parity of k' , see Chapter 1 of [7]. Hence, by considering O and the $q-1$ q -arcs whose planes contain l , we see that the number of tangents of K through N is at least $q+1+q-1=2q$. As K is complete we have $2q \leq t$, so $k \leq q^2 + q + 2 - 2q = q^2 - q + 2$, a contradiction.

Consequently $k \leq q^2 - q + 3$.

(B) Some tangent $l_i, 1 \leq i \leq t$, is contained in at least two planes having at most three points in common with K .

First we will prove that $k \leq q^2 - q + 5$. For $k = q^2 - q + 5$ and $k = q^2 - q + 4$ a contradiction will be obtained; the case $k = q^2 - q + 4$ will be subdivided in two cases. Hence it follows that also in Case (B) we have $k \leq q^2 - q + 3$.

Counting the points of K on the $q+1$ planes containing l_i gives

$$(8) \quad k - 1 \leq 2 \cdot 2 + (q-1)q = q^2 - q + 4.$$

So $k \leq q^2 - q + 5$.

(B.1) First, assume $k = q^2 - q + 5$. Then two planes Π_1, Π_2 containing l_i intersect K in 3 points, while the remaining planes $\Pi_3, \Pi_4, \dots, \Pi_{q+1}$ containing l_i intersect K in $q+1$ points. Let l be a tangent of K at P in Π_1 , distinct from l_i . Any plane ζ containing l , with $\zeta \neq \Pi_1$, intersects each $(q+1)$ -arc $\Pi_i \cap K, i = 3, 4, \dots, q+1$, in exactly two points. Hence $|\zeta \cap K| \geq q$. Considering the lines $\zeta \cap \Pi_2$, we see that exactly two of the planes ζ , say ζ_1 and ζ_2 , intersect K in $(q+1)$ -arcs O_1 and O_2 , while the $q-2$ other planes ζ , say $\zeta_3, \zeta_4, \dots, \zeta_q$, intersect K in a q -arc.

Let N_1 be the kernel of O_1 ; then $N_1 \in l$. The number of tangents of K containing N_1 is at least $q+1+q-2=2q-1$. As K is complete we have $2q-1 \leq t$, so $k \leq q^2 + q + 2 - 2q + 1 = q^2 - q + 3$, a contradiction.

(B.2) Next, assume $k = q^2 - q + 4$. Then, considering all planes containing l_i , there are two cases to consider.

(B.2.1) Two planes Π_1, Π_2 containing l_i intersect K in three points, the plane Π_3 containing l_i intersects K in q points, and the remaining planes $\Pi_4, \Pi_5, \dots, \Pi_{q+1}$ containing l_i intersect K in $q+1$ points. Let l be a tangent of K at P in Π_1 ,

distinct from l_i . Any plane ζ containing l , distinct from Π_1 , intersects each $(q+1)$ -arc $\Pi_i \cap K$, $i = 4, 5, \dots, q+1$, in exactly two points; $q-1$ of these planes ζ intersect $\Pi_3 \cap K$ in exactly two points. So for at least $q-1$ of these planes ζ we have $|\zeta \cap K| \geq q$, and for all planes ζ we have $|\zeta \cap K| \geq q-1$.

Assume that for all q planes ζ we have $|\zeta \cap K| \geq q$. Let s be the number of planes ζ for which $|\zeta \cap K| = q$ and let u be the number of planes ζ for which $|\zeta \cap K| = q+1$. Then

$$(9) \quad s(q-1) + uq + 3 = q^2 - q + 4, s + u = q.$$

So $s(q-1) + (q-s)q + 3 = q^2 - q + 4$, hence $s = q-1$ and $u = 1$. Let ζ be the plane which intersects K in a $(q+1)$ -arc O , and let $N \in l$ be the nucleus of O . The number of tangents of K containing N is at least $q+1+q-1 = 2q$, so $k \leq q^2 + q + 2 - 2q = q^2 - q + 2$, a contradiction.

So we may assume that for exactly $q-1$ planes ζ we have $|\zeta \cap K| \geq q$ and that for exactly one plane ζ we have $|\zeta \cap K| = q-1$. Assume that for s planes ζ we have $|\zeta \cap K| = q$, and that for u planes ζ we have $|\zeta \cap K| = q+1$. Then

$$(10) \quad s(q-1) + uq + q - 2 + 3 = q^2 - q + 4, s + u = q - 1.$$

So $s(q-1) + (q-1-s)q + q + 1 = q^2 - q + 4$, hence $s = q-3$ and $u = 2$. Let ζ_1, ζ_2 be the planes containing l which intersect K in $(q+1)$ -arcs O_1, O_2 , let N_1, N_2 be the nuclei of O_1, O_2 , and let $\Pi_1 \cap K = \{P, P_1, P_2\}$. Assume first that $N_1 \notin P_1P_2$. Then the number of tangents of K containing N_1 is at least $q+1+q-3+2 = 2q$, so $k \leq q^2 - q + 2$ a contradiction. Similarly if $N_2 \notin P_1P_2$. Hence we may assume that $N_1 = N_2 = P_1P_2 \cap l$. Then the number of tangents of K through N_1 is at least $q+1+q+q-3 = 3q-2$, so $k \leq q^2 + q + 2 - 3q + 2 = q^2 - 2q + 4$, again a contradiction.

(B.2.2) One plane Π_1 containing l_i intersects K in three points, and one plane Π_2 containing l_i intersects K in two points. Consequently the other $q-1$ planes $\Pi_3, \Pi_4, \dots, \Pi_{q+1}$ containing l_i intersect K in $q+1$ points. Let l be a tangent of K at P in Π_1 , distinct from l_i . Any plane ζ containing l , distinct from Π_1 , intersects each $(q+1)$ -arc $\Pi_i \cap K$, with $i = 3, 4, \dots, q+1$, in exactly two points. As $k = q^2 - q + 4$ it easily follows that for $q-1$ of these planes ζ we have $|\zeta \cap K| = q$, while for the remaining plane ζ we have $|\zeta \cap K| = q+1$.

Let $\tilde{\zeta}$ be the plane containing l which intersects K in a $(q+1)$ -arc O , and let N be the nucleus of O . The number of tangents of K containing N is at least $q+1+q-1 = 2q$, so $k \leq q^2 - q + 2$, again a contradiction.

(C) Some tangent l_i , with $1 \leq i \leq t$, is contained in no plane having at most three points in common with K .

First we will prove that $k \leq q^2 - q + 5$. A contradiction will be obtained for $k \in \{q^2 - q + 5, q^2 - q + 4\}$; for $k = q^2 - q + 4$ two cases have to be considered. Hence again $k \leq q^2 - q + 3$.

Then $|\Pi_j \cap K| \geq q-1$ for each plane Π_j containing l_i , with $j = 1, 2, \dots, q+1$. The arc $\Pi_j \cap K$ of Π_j can be completed to a $(q+2)$ -arc of Π_j ; see Theorem 1.2. This $(q+2)$ -arc meets l_i in points P, P_j . As there are $q+1$ points P_j and $|l_i \setminus \{P\}| = q$,

two of the points P_j coincide, say $P_1 = P_2$. The number of tangents of K containing P_1 is at least $2(q-2) + 1 = 2q-3$, so $k \leq q^2 - q + 5$.

Now we make some observations on $(q-1)$ -arcs of $\text{PG}(2, q)$, q even. Let \tilde{K} be any $(q-1)$ -arc of $\text{PG}(2, q)$, q even, and let \tilde{l} be a tangent of \tilde{K} at $\tilde{P} \in \tilde{K}$. Let \tilde{C} be the unique $(q+2)$ -arc which contains \tilde{K} ; see Theorem 1.2. Put $\tilde{C} \cap \tilde{l} = \{\tilde{P}, \tilde{N}\}$. Then it is easy to see that exactly $q-2$ points of $\tilde{l} \setminus \{\tilde{P}, \tilde{N}\}$ are on exactly three tangents of \tilde{K} , and that exactly one point \tilde{R} of $\tilde{l} \setminus \{\tilde{P}, \tilde{N}\}$ is on exactly one tangent of \tilde{K} ; also, $\tilde{R} = \tilde{l} \cap \tilde{N}'\tilde{N}''$, with $\{\tilde{N}, \tilde{N}', \tilde{N}''\} \cup \tilde{K} = \tilde{C}$.

(C.1) **First, assume** $k = q^2 - q + 5$. Then $\Pi_1 \cap K$ and $\Pi_2 \cap K$ are $(q-1)$ -arcs of Π_1 and Π_2 . Let r be the number of $(q-1)$ -arcs $\Pi_j \cap K$, let s be the number of q -arcs $\Pi_j \cap K$ and let u be the number of $(q+1)$ -arcs $\Pi_j \cap K$. Then

$$(11) \quad r(q-2) + s(q-1) + uq + 1 = q^2 - q + 5, r + s + u = q + 1, \text{ with } r \geq 2.$$

So $r(q-2) + s(q-1) + (q+1-r-s)q + 1 = q^2 - q + 5$, hence $2r + s = 2q - 4$, with $r \geq 2$. If $s \geq 1$, then we have an extra tangent of K containing P_1 , so $k \leq q^2 - q + 4$, a contradiction. Hence $s = 0, r = q - 2, u = 3$.

As the number of tangents of K containing P_1 is exactly $2q - 3$, the nuclei of the three $(q+1)$ -arcs $\Pi_j \cap K$ are distinct from P_1 . Let N be one of these nuclei. Also, P_1 is on exactly one tangent of each of the $q-4$ $(q-1)$ -arcs $\Pi_j \cap K$, distinct from the $(q-1)$ -arcs $\Pi_1 \cap K, \Pi_2 \cap K$. So N is on at least three tangents of each of these $q-4$ $(q-1)$ -arcs $\Pi_j \cap K$. Hence the number of tangents of K containing N is at least $2(q-4) + q + 1 = 3q - 7 > 2q - 3$, a contradiction.

(C.2) **Finally, assume that** $k = q^2 - q + 4$. We have to consider two cases depending of the sizes of $\Pi_1 \cap K$ and $\Pi_2 \cap K$.

(C.2.1) **First, assume that $\Pi_1 \cap K$ and $\Pi_2 \cap K$ are $(q-1)$ -arcs.** The tangents of K containing P_1 are the tangents of $\Pi_1 \cap K$ and $\Pi_2 \cap K$ containing P_1 , and one extra tangent l . Assume that l is a tangent of $\Pi_3 \cap K$. If $\Pi_3 \cap K$ is a $(q+1)$ -arc O , then P_1 is the nucleus of O , so there arise q extra tangents, a contradiction; if $\Pi_3 \cap K$ is a $(q-1)$ -arc K' , then P_1 is contained in at least three tangents of K' , again a contradiction. Hence $\Pi_3 \cap K$ is a q -arc. Also, $\Pi_j \cap K$, with $j = 4, 5, \dots, q+1$, cannot be a q -arc. Let r be the number of $(q-1)$ -arcs $\Pi_j \cap K$, and let u be the number of $(q+1)$ -arcs $\Pi_j \cap K$. Then

$$(12) \quad r(q-2) + uq + q - 1 + 1 = q^2 - q + 4, r + u + 1 = q + 1.$$

So $r(q-2) + (q-r)q + q = q^2 - q + 4$, hence $r = q - 2$ and $u = 2$. Let O_1, O_2 be the $(q+1)$ -arcs $\Pi_j \cap K$, and let N_1, N_2 be the nuclei of O_1, O_2 . Then $N_i \neq P_1, i = 1, 2$. Also P_1 is contained in exactly one tangent of each of the $q-4$ $(q-1)$ -arcs $\Pi_j \cap K$, with $j \neq 1, 2$. Hence the number of tangents of K containing N_1 is at least $2(q-4) + q + 1 = 3q - 7 > 2q - 2$, clearly a contradiction.

(C.2.2) **Consequently, we may assume that $\Pi_1 \cap K$ is a $(q-1)$ -arc and that $\Pi_2 \cap K$ is a q -arc.** Let r be the number of $(q-1)$ -arcs $\Pi_j \cap K$, let s be the number of q -arcs $\Pi_j \cap K$ and let u be the number of $(q+1)$ -arcs $\Pi_j \cap K$. Then

$$(13) \quad r(q-2) + s(q-1) + uq + 1 = q^2 - q + 4, r + s + u = q + 1, r \geq 1, s \geq 1.$$

So $2r + s = 2q - 3$, $r \geq 1$, $s \geq 1$. Clearly, $s = 1$, as otherwise we have an extra tangent containing P_1 , and then $k < q^2 - q + 4$. Hence $r = q - 2$, $s = 1$, $u = 2$. The nuclei of the two $(q + 1)$ -arcs $\Pi_j \cap K$ are distinct from P_1 . Let N be one of these nuclei. Also, P_1 is on exactly one tangent of each of the $q - 3$ $(q - 1)$ -arcs $\Pi_j \cap K$ distinct from $\Pi_1 \cap K$. So N is on at least three tangents of each of these $q - 3$ $(q - 1)$ -arcs $\Pi_j \cap K$. Consequently the number of tangents of K containing N is at least $2(q - 3) + q + 1 = 3q - 5 > 2q - 2$, a final contradiction. ■

3. MAIN THEOREM

Theorem 3.1.

$$(14) \quad m'_2(3, q) < q^2 - (\sqrt{5} - 1)q + 5, q \text{ even}, q \geq 8.$$

$$(15) \quad m'_2(3, 4) = 14.$$

Proof By [8] we have $m'_2(3, 4) = 14$, and by Theorem 2.1 we have $m'_2(3, 8) \leq 59$, which proves Theorem 3.1 for $q = 8$. So from now on we assume $q > 8$.

Let K be a complete k -cap in $\mathbf{PG}(3, q)$, q even, $q > 8$, and $k < q^2 + 1$. Let Π be a plane of $\mathbf{PG}(3, q)$ for which

$$(16) \quad 5 \leq |\Pi \cap K| \leq q - 3.$$

Let $f(X) = q(q + 2 - X)X$. Then by Lemma 1.5 of Chao

$$(17) \quad t(t - 1) \geq f(5) = f(q - 3) = 5q(q - 3).$$

So

$$(18) \quad t \geq \frac{1 + \sqrt{1 + 20q(q - 3)}}{2}.$$

Put $\frac{1 + \sqrt{1 + 20q(q - 3)}}{2} \geq \sqrt{5}q - \alpha$, that is,

$$(19) \quad \sqrt{1 + 20q(q - 3)} \geq 2\sqrt{5}q - 2\alpha - 1.$$

For $\alpha \leq \sqrt{5}q - (1/2)$ this is equivalent to

$$(20) \quad 1 + 20q(q - 3) \geq 20q^2 + 4\alpha^2 + 1 - 8\alpha\sqrt{5}q - 4\sqrt{5}q + 4\alpha,$$

or

$$(21) \quad 0 \geq 4\alpha^2 + \alpha(-8\sqrt{5}q + 4) + 60q - 4\sqrt{5}q,$$

or

$$(22) \quad 0 \geq \alpha^2 + \alpha(-2\sqrt{5}q + 1) + 15q - \sqrt{5}q.$$

Put $\alpha = 3$. Then there arises $0 \geq 9 + 3(-2\sqrt{5}q + 1) + 15q - \sqrt{5}q$, that is, $0 \geq 12 + 15q - 7\sqrt{5}q$. This inequality is satisfied for $q > 16$.

Hence for $q > 16$ we have $t \geq \sqrt{5}q - 3$, and so,

$$(23) \quad k \leq q^2 + q + 2 - \sqrt{5}q + 3,$$

that is,

$$(24) \quad k \leq q^2 + (1 - \sqrt{5})q + 5.$$

For $q = 16$ it follows from (18) that $t > 32$ and so $k \leq 241$, which is equivalent to $k \leq q^2 + (1 - \sqrt{5})q + 5$ with $q = 16$.

From now on suppose that either $|\Pi \cap K| \leq 4$ or $|\Pi \cap K| \geq q - 2$ for any plane Π of $\text{PG}(3, q)$. Let l_1, l_2, \dots, l_t be the t tangents of K containing the point $P \in K$. Assume, by way of contradiction, that $k > q^2 + (1 - \sqrt{5})q + 5$. We consider three cases depending on the number of planes containing l_i and intersecting K in at most 4 points. In each case a contradiction will be obtained .

(A) Assume, by way of contradiction, that each l_i is contained in exactly one plane Π_{l_i} for which $|\Pi_{l_i} \cap K| \leq 4$, with $i = 1, 2, \dots, t$.

(A.1) Assume that there is exactly one plane Π through P with $|\Pi \cap K| \leq 4$. Then for $i = 1, 2, \dots, t$ we have $\Pi_{l_i} = \Pi$. So $t \leq q + 1$, hence $k \geq q^2 + 1$, a contradiction.

(A.2) There are at least two planes Π_1, Π_2 through P such that $|\Pi_i \cap K| \leq 4, i = 1, 2$. Then $|\Pi_1 \cap \Pi_2 \cap K| = 2$. Consequently $t \geq 2(q - 2)$, and so $k \leq q^2 + q + 2 - 2q + 4 = q^2 - q + 6$.

The plane Π_1 intersects K in a m -arc, $m \leq 4$, and contains at least $q - 2$ tangents of K at P . Let $P_1 \in (K \cap \Pi_1) \setminus P$ and assume that PP_1 is contained in α planes Π with $|\Pi \cap K| \leq 4$. Then $t \geq \alpha(q - 2)$, so $k \leq q^2 + (1 - \alpha)q + 2 + 2\alpha$. Consequently

$$(25) \quad q^2 + (1 - \alpha)q + 2 + 2\alpha > q^2 + (1 - \sqrt{5})q + 5,$$

or

$$(26) \quad (\sqrt{5} - \alpha)q + 2\alpha - 3 > 0$$

This gives a contradiction for $\alpha > 2$ with $q > 8$. So PP_1 is contained in at most two planes intersecting K in at most four points.

Assume, by way of contradiction, that for some plane Π of $\text{PG}(3, q)$ we have $\Pi \cap K = \{P\}$. As there are at least two planes Π, Π' through P intersecting K in at most four points, we have $|\Pi \cap \Pi' \cap K| = 2$ and so $|\Pi \cap K| \geq 2$, a contradiction.

Let $\text{PG}(2, q)$ be a plane of $\text{PG}(3, q)$ not containing P and let σ be the projection of $\text{PG}(3, q) \setminus \{P\}$ from P onto $\text{PG}(2, q)$. Further, let \mathcal{P} be the set of all images under σ of all points of $K \setminus \{P\}$ contained in planes Π , with $P \in \Pi$, for which $|\Pi \cap K| \leq 4$, and let \mathcal{B} be the set of all images under σ of the sets $\Pi \setminus \{P\}$. Then there arises an incidence structure $(\mathcal{P}, \mathcal{B})$ of points and lines for which

- (1) $|\mathcal{B}| \geq 2$,
- (2) any two distinct lines in \mathcal{B} have exactly one point in common,
- (3) each point is contained in at most two lines,
- (4) each line contains at most three points and at least one point.

It follows easily that $2 \leq |\mathcal{B}| \leq 4$. For each value of $|\mathcal{B}|$ we will find a contradiction.

(α) $|\mathcal{B}| = 4$

Then $t = 4(q - 2)$, so $k = q^2 + q + 2 - 4q + 8 = q^2 - 3q + 10$. Hence $q^2 - 3q + 10 > q^2 + (1 - \sqrt{5})q + 5$, or $5 > (4 - \sqrt{5})q$, a contradiction as $q > 8$.

(β) $|\mathcal{B}| = 3$

If $|\mathcal{P}| = 3$, then $t = 3(q-1)$, so $k = q^2 - 2q + 5$. Hence $q^2 - 2q + 5 > q^2 + (1 - \sqrt{5})q + 5$, or $(3 - \sqrt{5})q < 0$, a contradiction.

If $|\mathcal{P}| = 4$, then $t = 2(q-1) + q - 2$, so $k = q^2 - 2q + 6$. Hence $q^2 + (1 - \sqrt{5})q + 5 < q^2 - 2q + 6$, or $(3 - \sqrt{5})q - 1 < 0$, a contradiction.

If $|\mathcal{P}| = 5$, then $t = q - 1 + 2(q-2)$, so $k = q^2 - 2q + 7$. Hence $q^2 + (1 - \sqrt{5})q + 5 < q^2 - 2q + 7$, or $(3 - \sqrt{5})q < 2$, a contradiction.

If $|\mathcal{P}| = 6$, then $t = 3(q-2)$, so $k = q^2 - 2q + 8$. Hence $q^2 + (1 - \sqrt{5})q + 5 < q^2 - 2q + 8$, or $(3 - \sqrt{5})q < 3$, a contradiction.

(γ) $|\mathcal{B}| = 2$

By Theorem 2.1 we may assume that $k \leq q^2 - q + 3$.

If $|\mathcal{P}| = 1$, then $t = 2q$, so $k = q^2 - q + 2$.

If $|\mathcal{P}| = 2$, then $t = 2q - 1$, so $k = q^2 - q + 3$.

If $|\mathcal{P}| = 3$, then $t = 2q - 2$, so $k = q^2 - q + 4$, a contradiction.

If $|\mathcal{P}| = 4$, then $t = 2q - 3$, so $k = q^2 - q + 5$, a contradiction.

If $|\mathcal{P}| = 5$, then $t = 2q - 4$, so $k = q^2 - q + 6$, a contradiction.

Hence the cases $k = q^2 - q + 2$ and $k = q^2 - q + 3$ have still to be considered.

($\gamma.1$) $k = q^2 - q + 2$

On K there are two points P, P_1 such that PP_1 is contained in two planes Π_1, Π_2 intersecting K in just $\{P, P_1\}$, and in $q-1$ planes $\Pi_3, \Pi_4, \dots, \Pi_{q+1}$ intersecting K in a $(q+2)$ -arc.

Let $P' \in (\Pi_3 \cap K) \setminus \{P, P_1\}$ and let l be a tangent of K at P' . Assume, by way of contradiction, that each plane containing l intersects K in a m -arc with $m > 4$, so $m \geq q-2$. These m -arcs K'_i , with $i = 1, 2, \dots, q+1$, are extendable to $(q+2)$ -arcs C_i . Let $C_i \cap l = \{N_i, P'\}$, $i = 1, 2, \dots, q+1$. At least two of the points N_1, N_2, \dots, N_{q+1} coincide, say $N_1 = N_2$. A plane Π' containing l , but not containing P nor P_1 , intersects each of the $(q+2)$ -arcs $\Pi_i \cap K$, with $i = 3, 4, \dots, q+1$, in either 0 or 2 points; so $|\Pi' \cap K|$ is even. A plane Π' containing l and either P or P_1 intersects K in q points. Hence each plane containing l intersects K in a m -arc, with m even. Counting tangents of K containing N_1 , we obtain at least $2(q-3) + 1 + q - 1 = 3q - 6$ tangents. So $k \leq q^2 + q + 2 - 3q + 6 = q^2 - 2q + 8$, a contradiction for $q > 8$. We conclude that there is a plane Π' containing l with $|\Pi' \cap K| \leq 4$.

Assume, by way of contradiction, that l is contained in at least two planes Π', Π'' with $|\Pi' \cap K| \leq 4, |\Pi'' \cap K| \leq 4$. Then, by a previous argument, these intersections have an even number of points and so $|\Pi' \cap K| \in \{2, 4\}$ and $|\Pi'' \cap K| \in \{2, 4\}$. Now we count the points of K in planes containing l , and obtain $k \leq (q-1)(q-1) + 7 = q^2 - 2q + 8$, a contradiction for $q > 8$.

Hence l is contained in exactly one plane Π' for which $|\Pi' \cap K| \leq 4$. It follows that the roles of P and P' may be interchanged.

Let l' be a second tangent of K containing P' , with $l' \not\subset \Pi'$. Let $\tilde{K} = K \cap \Pi_3, \Pi' \cap \tilde{K} = \{P', P'_1\}$. If $P'_1 \in \{P, P_1\}$, then $|\Pi' \cap K| = q$, a contradiction. Hence $P'_1 \notin \{P, P_1\}$. With P' there corresponds an incidence structure $(\mathcal{P}', \mathcal{B}')$ of points and

lines. As $k = q^2 - q + 2$, we necessarily have $|\mathcal{P}'| = 1$ and $|\mathcal{B}'| = 2$. Hence $\Pi' \cap K = \{P', P'_1\}$. If $\tilde{\Pi}'$ is the unique plane containing l' and intersecting K in at most 4 points, then $\tilde{\Pi}' \cap K = \{P', P'_1\}$. Also, the roles of P and P'_1 , P' and P'_1 , P and P_1 can be interchanged.

Interchanging Π_3 and Π_i , $i \in \{3, 4, \dots, q+1\}$, and interchanging P' with any point of $(\Pi_i \cap K) \setminus \{P, P_1\}$, we see that K is partitioned into $(q^2 - q + 2)/2$ pairs, where each pair is contained in two planes intersecting K in that pair and in $q-1$ planes intersecting K in a $(q+2)$ -arc. Any other plane contains either 0 or q points of K . Each point Q of K is contained in $2q$ tangents; the two planes on Q intersecting K in two points each contain q of these tangents.

Now we count the planes intersecting K in a $(q+2)$ -arc, and obtain

$$(27) \quad \frac{q^2 - q + 2}{2} \cdot (q-1) / \frac{q+2}{2}.$$

Hence $q+2 \mid (q^2 - q + 2)(q-1)$, so $q+2 \mid 24$, that is $q \in \{2, 4\}$, a contradiction.

($\gamma.2$) $k = q^2 - q + 3$

Then on K there are points P, P_1 such that PP_1 is contained in two planes Π_1, Π_2 with $\Pi_1 \cap K = \{P, P_1\}$, $\Pi_2 \cap K = \{P, P_1, P_2\}$, and in $q-1$ planes $\Pi_3, \Pi_4, \dots, \Pi_{q+1}$ intersecting K in a $(q+2)$ -arc.

Let $P' \in (\Pi_3 \cap K) \setminus \{P, P_1\}$ and let l be a tangent of K at P' . Assume, by way of contradiction, that each plane containing l intersects K in a m -arc with $m > 4$, so $m \geq q-2$. These m -arcs K'_i , with $i = 1, 2, \dots, q+1$, are extendable to $(q+2)$ -arcs C_i . Let $C_i \cap l = \{N_i, P'\}$, $i = 1, 2, \dots, q+1$. At least two of the points N_1, N_2, \dots, N_{q+1} coincide, say $N_1 = N_2$. A plane Π' containing l , but not containing P nor P_1 , intersects each of the $(q+2)$ -arcs $\Pi_i \cap K$, with $i = 3, 4, \dots, q+1$, in either 0 or 2 points. So if $P_2 \notin \Pi'$, then $|\Pi' \cap K|$ is even. A plane Π' containing l and either P or P_1 , but not P_2 , intersects K in q points. Hence q planes containing l intersect K in a m -arc, with m even. Counting tangents of K containing N_1 , we obtain at least $2(q-3) + 1 + q - 2 = 3q - 7$ tangents. So $k \leq q^2 + q + 2 - 3q + 7 = q^2 - 2q + 9$, a contradiction for $q > 8$. We conclude that there is a plane Π' containing l with $|\Pi' \cap K| \leq 4$.

Assume, by way of contradiction, that l is contained in at least two planes Π', Π'' with $|\Pi' \cap K| \leq 4, |\Pi'' \cap K| \leq 4$. Now we count the points of K in planes containing l , and obtain $k \leq q^2 - 2q + 9$, a contradiction for $q > 8$.

Hence l is contained in exactly one plane Π' for which $|\Pi' \cap K| \leq 4$. As all tangents of K at P_1 are contained in $\Pi_1 \cup \Pi_2$, it follows that each tangent of K at P_1 is contained in exactly one plane intersecting K in at most 4 points. Hence all points of $K \setminus \{P_2\}$ play the same role.

Let l' be a second tangent of K containing P' , with $l' \not\subset \Pi'$. Let $K \cap \Pi_3 = \tilde{K}$, $\Pi' \cap \tilde{K} = \{P', P'_1\}$. If $P'_1 \in \{P, P_1\}$, then $|\Pi' \cap K| \geq q$, a contradiction. Hence $P'_1 \notin \{P, P_1\}$. With P' there corresponds an incidence structure $(\mathcal{P}', \mathcal{B}')$ of points and lines (see first part of (A)).

As $k = q^2 - q + 3$, we necessarily have $|\mathcal{P}'| = 2$ and $|\mathcal{B}'| = 2$. Hence $|\Pi' \cap K| \in \{2, 3\}$ and $\Pi' \cap K \supset \{P', P'_1\}$. Let $\tilde{\Pi}'$ be the unique plane containing l' and intersecting K in at most 4 points, and let $\tilde{\Pi}' \cap \tilde{K} = \{P', \tilde{P}'_1\}$. If $P'_1 \neq \tilde{P}'_1$, then by the structure

of $(\mathcal{P}', \mathcal{B}')$ we have $\{P'_1, \tilde{P}'_1\} \subset \Pi'$, clearly a contradiction. Hence $P'_1 = \tilde{P}'_1$, and so $\{P', P'_1\} \subset \tilde{\Pi}' \cap K$.

Without loss of generality we may assume that $\Pi' \cap K = \{P', P'_1, P'_2\}$ and $\tilde{\Pi}' \cap K = \{P', P'_1\}$. As $|\Pi' \cap K|$ is odd, the set $\Pi' \cap K$ has to contain the point P_2 . Consequently $P_2 = P'_2$.

Interchanging Π_3 and Π_i , $i \in \{3, 4, \dots, q+1\}$, and interchanging P' with any point of $(\Pi_i \cap K) \setminus \{P, P_1\}$, we see that $K \setminus \{P_2\}$ is partitioned into $(q^2 - q + 2)/2$ pairs, where each pair is contained in one plane intersecting K in that pair, in one plane intersecting K in that pair together with P_2 , and in $q - 1$ planes intersecting K in a $(q + 2)$ -arc. Any other plane contains 0, 1, q or $q + 1$ points of K .

Now we count the planes intersecting K in a $(q + 2)$ -arc and obtain

$$(28) \quad \frac{q^2 - q + 2}{2} \cdot (q - 1) / \frac{q + 2}{2}.$$

Hence $q + 2|(q^2 - q + 2)(q - 1)$, so $q + 2|24$, that is $q \in \{2, 4\}$, a final contradiction.

We conclude that there is some tangent l_i containing P , with $i \in \{1, 2, \dots, t\}$, which is contained in exactly $\theta > 1$ planes having at most 4 points in common with K .

(B) Assume, by way of contradiction, that some tangent l of K is contained in no plane intersecting K in at most 4 points.

Hence each plane Π_i containing l satisfies $|\Pi_i \cap K| \geq q - 2$, with $i = 1, 2, \dots, q + 1$. By Theorem 1.2 the arc $\Pi_i \cap K$ can be extended to a $(q + 2)$ -arc C_i ; let $C_i \cap l = \{N_i, P\}$ with $l \cap K = \{P\}$. For at least two planes Π_i , say Π_1 and Π_2 , we have $N_1 = N_2$.

(B.1) First we prove that N_1 is on a tangent of K not in $\Pi_1 \cup \Pi_2$; clearly N_1 is on at least $2q - 5$ tangents of K contained in $\Pi_1 \cup \Pi_2$. Assume the contrary. Then for any plane $\Pi_i \notin \{\Pi_1, \Pi_2\}$, the arc $\Pi_i \cap K$ must have an odd number of points. So $\Pi_i \cap K$ either is a $(q - 1)$ -arc or a $(q + 1)$ -arc, $i \in \{3, 4, \dots, q + 1\}$. Also, $N_i \neq N_1$ for $i = 3, 4, \dots, q - 1$. If $\Pi_i \cap K$ is a $(q - 1)$ -arc and $C_i \setminus (\Pi_i \cap K) = \{N_i, N'_i, N''_i\}$, $i \in \{3, 4, \dots, q + 1\}$, then $N_1 \in N'_i N''_i$, as otherwise $N_1 N'_i$ and $N_1 N''_i$ are tangents of $\Pi_i \cap K$.

Let r be the number of planes Π_i , with $i \neq 1, 2$, for which $\Pi_i \cap K$ is a $(q - 1)$ -arc, and let s be the number of planes Π_i , with $i \neq 1, 2$, for which $\Pi_i \cap K$ is a $(q + 1)$ -arc. The number of points of K is at least

$$(29) \quad r(q - 2) + sq + 2(q - 3) + 1, \text{ with } r + s = q - 1.$$

As K is complete, by Theorem 2.1

$$(30) \quad r(q - 2) + (q - 1 - r)q + 2(q - 3) + 1 \leq q^2 - q + 3,$$

so

$$(31) \quad r \geq q - 4.$$

We may assume that $\Pi_3 \cap K$ is a $(q - 1)$ -arc. The number of tangents of K containing N_3 is at least

$$(32) \quad q - 1 + 2(r - 1) \geq q - 1 + 2q - 10 = 3q - 11.$$

Hence

$$(33) \quad k \leq q^2 + q + 2 - 3q + 11 = q^2 - 2q + 13.$$

So

$$(34) \quad q^2 - 2q + 13 > q^2 + (1 - \sqrt{5})q + 5,$$

a contradiction for $q > 8$.

Consequently N_1 is on a tangent l' of K not in $\Pi_1 \cup \Pi_2$.

(B.2) Now we consider all planes Π'_i containing the tangent l' , with $i = 1, 2, \dots, q+1$. We will show that:

(a) For each plane Π'_i such that $|\Pi'_i \cap K| \geq q-2$ the point N_1 does not extend the arc $\Pi'_i \cap K$.

(b) For each i we have $|\Pi'_i \cap K| \geq q-2$.

(a) Let $|\Pi_1 \cap K| = \alpha, q-2 \leq \alpha \leq q+1, |\Pi_2 \cap K| = \beta, q-2 \leq \beta \leq q+1$. Then N_1 is contained in at least $\alpha + \beta$ tangents of K . Now we consider all planes Π'_i containing the tangent l' , with $i = 1, 2, \dots, q+1$. Assume, by way of contradiction, that $m = |\Pi'_i \cap K| \geq q-2$ and that the $(q+2)$ -arc C'_i extending $\Pi'_i \cap K$ intersects l' in $\{N_1, P'\}$, with $l' \cap K = \{P'\}, i \in \{1, 2, \dots, q+1\}$. Then the number of tangents of K containing N_1 is at least

$$(35) \quad \alpha + \beta + m - 3 \geq 2q - 4 + m - 3 \geq 3q - 9.$$

Hence

$$(36) \quad k \leq q^2 + q + 2 - 3q + 9 = q^2 - 2q + 11.$$

So $q^2 - 2q + 11 > q^2 + (1 - \sqrt{5})q + 5$, a contradiction. Consequently for $|\Pi'_i \cap K| \geq q-2$ we have $N_1 \notin C'_i, i \in \{1, 2, \dots, q+1\}$.

(b) Next, assume by way of contradiction that for at least one plane Π'_i containing l' , say Π'_1 , we have $|\Pi'_1 \cap K| \leq 4$. Let Π'_2 be the plane ll' . Now we count the points of K in the planes Π'_i , with $i = 1, 2, \dots, q+1$. Let

θ_1 be the number of planes $\Pi'_i, i \in \{3, 4, \dots, q+1\}$, containing a tangent of $\Pi_1 \cap K$ through N_1 and a tangent of $\Pi_2 \cap K$ through N_1 ,

θ_2 be the number of planes $\Pi'_i, i \in \{3, 4, \dots, q+1\}$, containing a tangent of $\Pi_1 \cap K$ through N_1 , but no tangent of $\Pi_2 \cap K$ through N_1 ,

θ_3 be the number of planes $\Pi'_i, i \in \{3, 4, \dots, q+1\}$, containing a tangent of $\Pi_2 \cap K$ through N_1 , but no tangent of $\Pi_1 \cap K$ through N_1 ,

θ_4 be the number of planes $\Pi'_i, i \in \{3, 4, \dots, q+1\}$, containing no one of the tangents of $\Pi_1 \cap K$ or $\Pi_2 \cap K$ through N_1 .

Then, as $N_1 \notin C'_i$ for $|\Pi'_i \cap K| \geq q-2$, we have

$$(37) \quad k \leq 4 + q - 1 + \theta_1(q-2) + \theta_2(q-1) + \theta_3(q-1) + \theta_4q, \text{ with } 2 + \theta_1 + \theta_2 + \theta_3 + \theta_4 = q+1.$$

Hence

$$(38) \quad k \leq q(\theta_1 + \theta_2 + \theta_3 + \theta_4) - (2\theta_1 + \theta_2 + \theta_3) + q + 3,$$

so

$$(39) \quad k \leq q(q-1) - (2\theta_1 + \theta_2 + \theta_3) + q + 3.$$

Now we have

$$\theta_1 + \theta_2 \geq |\Pi_1 \cap K| - 2 \geq q - 4,$$

$$\theta_1 + \theta_3 \geq |\Pi_2 \cap K| - 2 \geq q - 4.$$

Hence

$$(40) \quad k \leq q(q-1) - 2q + 8 + q + 3 = q^2 - 2q + 11.$$

So $q^2 - 2q + 11 > q^2 + (1 - \sqrt{5})q + 5$, a contradiction.

Hence no plane Π'_i containing l' intersects K in a m -arc, with $m \leq 4$, $1 \leq i \leq q+1$. Consequently, for each plane Π'_i containing l' we have $|\Pi'_i \cap K| \geq q-2$. Also, we know that the $(q+2)$ -arc C'_i extending $\Pi'_i \cap K$ does not contain N_1 , with $i = 1, 2, \dots, q+1$.

(B.3) A final contradiction will be obtained by considering the possible intersections $\Pi'_i \cap K$, $i = 1, 2, \dots, q+1$. It is easy to see that at least $q-6$ planes Π'_i containing l' intersect K in a m -arc having at least 3 tangents containing N_1 ; these planes are the planes containing l' passing through distinct tangents of $\Pi_1 \cap K$ and $\Pi_2 \cap K$ containing N_1 . For any such plane Π'_i the arc $\Pi'_i \cap K$ is either a $(q-1)$ -arc or a $(q-2)$ -arc. Let

θ'_1 be the number of planes Π'_i , with $\Pi'_i \neq ll'$, containing a tangent of $\Pi_1 \cap K$ through N_1 , a tangent of $\Pi_2 \cap K$ through N_1 , where $\Pi'_i \cap K$ is a $(q-1)$ -arc,

θ'_2 be the number of planes Π'_i , with $\Pi'_i \neq ll'$, containing a tangent of $\Pi_1 \cap K$ through N_1 , a tangent of $\Pi_2 \cap K$ through N_1 , where $\Pi'_i \cap K$ is a $(q-2)$ -arc.

Let $C'_i \cap l' = \{P', N'_i\}$, with $l' \cap K = \{P'\}$ and C'_i the $(q+2)$ -arc extending $\Pi'_i \cap K$, $i = 1, 2, \dots, q+1$. Then $N'_i \neq N_1$, $i = 1, 2, \dots, q+1$. We may assume that $N'_1 = N'_2$. Assume, by way of contradiction, that $N'_1 = N'_2 = N'_i$, with $i \in \{3, 4, \dots, q+1\}$. Then N'_1 is on at least $3(q-3) + 1$ tangents of K . So

$$(41) \quad k \leq q^2 + q + 2 - 3q + 8 = q^2 - 2q + 10.$$

Hence

$$(42) \quad q^2 + (1 - \sqrt{5})q + 5 < q^2 - 2q + 10,$$

that is,

$$(43) \quad (3 - \sqrt{5})q < 5,$$

clearly a contradiction. Hence we may assume that $N'_1 = N'_2, N'_3 = N'_4, N'_1 \neq N'_3, N'_i \notin \{N'_1, N'_3\}$ for $i = 5, 6, \dots, q+1$. At least $\theta'_1 - 4$ of the arcs $\Pi'_5 \cap K, \Pi'_6 \cap K, \dots, \Pi'_{q+1} \cap K$ are $(q-1)$ -arcs, say $\Pi'_5 \cap K, \Pi'_6 \cap K, \dots, \Pi'_{\theta'_1} \cap K$ are $(q-1)$ -arcs. The number of tangents of $\Pi'_i \cap K$ containing N'_j , with $j \in \{1, 3\}$, is either 1 or 3, with $i = 5, 6, \dots, \theta'_1$; if N'_j is contained in one tangent of $\Pi'_i \cap K$, then N'_u is contained in 3 tangents of $\Pi'_i \cap K$, with $\{j, u\} = \{1, 3\}$ and $i \in \{5, 6, \dots, \theta'_1\}$. So we may assume that at least $(\theta'_1 - 4)/2$ of the $(q-1)$ -arcs $\Pi'_i \cap K$, $i = 5, 6, \dots, \theta'_1$, have 3 tangents containing N'_1 . Counting the tangents of K through N'_1 , we obtain at least

$$(44) \quad 1 + (\theta'_1 - 4) + (\theta'_2 - 2) + 2(q-3)$$

tangents. As $\theta'_1 + \theta'_2 \geq q-6$, this number of tangents is at least $1 + q - 6 - 6 + 2q - 6 = 3q - 17$. Hence

$$(45) \quad k \leq q^2 + q + 2 - 3q + 17 = q^2 - 2q + 19.$$

So

$$(46) \quad q^2 + (1 - \sqrt{5})q + 5 < q^2 - 2q + 19,$$

or

$$(47) \quad (3 - \sqrt{5})q < 14,$$

a contradiction for $q > 16$.

If at least one of the arcs $\Pi'_1 \cap K, \Pi'_2 \cap K$ is a m -arc with $m > q - 2$, then (44) becomes

$$(48) \quad 1 + (\theta'_1 - 4) + (\theta'_2 - 1) + (q - 3) + (q - 2),$$

which is at least $3q - 15$. Hence $k \leq q^2 - 2q + 17$. For $q = 16$ this gives $k \leq 241$. But for $q = 16$ the inequality $k > q^2 + (1 - \sqrt{5})q + 5$ yields $k \geq 242$, a contradiction.

Finally we assume that $\Pi'_1 \cap K$ and $\Pi'_2 \cap K$ are $(q - 2)$ -arcs. Then at least $\theta'_1 - 2$ of the arcs $\Pi'_i \cap K$, with $i = 5, 6, \dots, q + 1$, are $(q - 1)$ -arcs, say $\Pi'_5 \cap K, \Pi'_6 \cap K, \dots, \Pi'_{\theta'_1 + 2} \cap K$. So at least $(\theta'_1 - 2)/2$ of the $(q - 1)$ -arcs $\Pi'_i \cap K$, with $i = 5, 6, \dots, \theta'_1 + 2$, have 3 tangents containing either N'_1 or N'_3 . First, assume that this is the case for N'_3 . If at least one of the arcs $\Pi'_3 \cap K, \Pi'_4 \cap K$ is a m -arc with $m > q - 2$, then the number of tangents of K containing N'_3 is at least

$$(49) \quad 1 + (\theta'_1 - 2) + (\theta'_2 - 1) + (q - 3) + (q - 2),$$

which is at least $3q - 13$. Hence $k \leq q^2 - 2q + 15$, and so $q^2 + (1 - \sqrt{5})q + 5 < q^2 - 2q + 15$, that is, $(3 - \sqrt{5})q < 10$, a contradiction. Hence the arcs $\Pi'_3 \cap K$ and $\Pi'_4 \cap K$ are $(q - 2)$ -arcs. Then the number of tangents of K containing N'_3 is at least

$$(50) \quad 1 + \theta'_1 + (\theta'_2 - 2) + 2(q - 3),$$

which is at least $3q - 13$. This yields again a contradiction. Consequently at least $(\theta'_1 - 2)/2$ of the $(q - 1)$ -arcs $\Pi'_i \cap K$, with $i = 5, 6, \dots, \theta'_1 + 2$, have 3 tangents containing N'_1 . But then in (44) $\theta'_1 - 4$ may be replaced by $\theta'_1 - 2$, yielding at least $3q - 15$ tangents of K containing N'_1 . Hence $k \leq q^2 - 2q + 17$, which is a final contradiction.

We conclude that each tangent l of K is contained in at least one plane intersecting K in at most four points.

(C) Assume, by way of contradiction, that there is a tangent l of K which is contained in at least two planes Π_1, Π_2 intersecting K in a m -arc, with $m \leq 4$.

Assume that $l \cap K = \{P\}$ and that $\Pi_1 \cup \Pi_2$ contains $2q + \delta$ tangents of K through P . We have $-5 \leq \delta \leq 1$.

(C.1) Here we will show that $2q + \delta$ is the total number of tangents of K containing P ; as a corollary it will follow that $k \in \{q^2 - q + 1, q^2 - q + 2, q^2 - q + 3\}$. Assume, by way of contradiction, that there is a tangent l' of K containing P with $l' \not\subset \Pi_1 \cup \Pi_2$. If $|l' \cap K| \leq 4$, then the number of tangents of K containing P is at least $2q + \delta + q - 3 = 3q + \delta - 3 \geq 3q - 8$, so $k \leq q^2 + q + 2 - 3q + 8 = q^2 - 2q + 10$. Hence

$$(51) \quad q^2 + (1 - \sqrt{5})q + 5 < q^2 - 2q + 10,$$

or $(3 - \sqrt{5})q < 5$, a contradiction. Now we consider all planes containing l' . By

(B) at least one of these planes intersects K in a m -arc, with $m \leq 4$. If at least two planes containing l' intersect K in at most 4 points, then P is contained in at least

$2q - 5 + 2(q - 5) + 1 = 4q - 14$ tangents of K . Hence $k \leq q^2 + q + 2 - 4q + 14 = q^2 - 3q + 16$, so

$$(52) \quad q^2 + (1 - \sqrt{5})q + 5 < q^2 - 3q + 16,$$

that is, $(4 - \sqrt{5})q < 11$, clearly a contradiction. Consequently exactly one plane Π' containing l' intersects K in at most 4 points. Now we count the points of K in the planes containing l' . Let

θ_1 be the number of planes, distinct from ll' and Π' , containing l' , containing a tangent of K in Π_1 and containing a tangent of K in Π_2 ,

θ_2 be the number of planes containing l' , distinct from Π' , containing a tangent of K in Π_1 and containing no tangent of K in Π_2 ,

θ_3 be the number of planes containing l' , distinct from Π' , containing a tangent of K in Π_2 and containing no tangent of K in Π_1 ,

θ_4 be the number of planes, distinct from Π' , containing l' and containing no tangent of K in Π_1 or Π_2 .

Then

$$(53) \quad k \leq 1 + (q - 1) + \theta_1(q - 2) + \theta_2(q - 1) + \theta_3(q - 1) + \theta_4q + 3,$$

with

$$(54) \quad \theta_1 + \theta_2 + \theta_3 + \theta_4 = q - 1 \text{ and } \theta_2 + \theta_3 + 2\theta_4 \leq 6.$$

So

$$(55) \quad k \leq q + 3 + (q - 1 - \theta_2 - \theta_3 - \theta_4)(q - 2) + \theta_2(q - 1) + \theta_3(q - 1) + \theta_4q,$$

that is,

$$(56) \quad k \leq q^2 - 2q + 5 + (\theta_2 + \theta_3 + 2\theta_4),$$

hence

$$(57) \quad k \leq q^2 - 2q + 5 + 6 = q^2 - 2q + 11.$$

Consequently

$$(58) \quad q^2 + (1 - \sqrt{5})q + 5 < q^2 - 2q + 11,$$

or $(3 - \sqrt{5})q < 6$, a contradiction.

It follows that $2q + \delta$ is the total number of tangents of K containing P and so $k = q^2 + q + 2 - 2q - \delta = q^2 - q + 2 - \delta$. As $k \leq q^2 - q + 3$ by Theorem 2.1, we have $-1 \leq \delta \leq 1$.

(C.2) Some further properties of K . Let l'' be any tangent of K not containing P and let $K \cap l'' = \{P'\}$. By (B) l'' is contained in a plane Π'' with $|\Pi'' \cap K| \leq 4$. There is a tangent n of K at P' not contained in Π'' . The tangent n is contained in a plane ρ with $|\rho \cap K| \leq 4$. Let $2q + \delta'$ be the number of tangents of K at P' in $\rho \cup \Pi''$. Then $\delta' \leq \delta$ and if $\rho \cap \Pi''$ is a tangent, then by the foregoing section we have $\delta' = \delta$. Assume, by way of contradiction, that $\rho \cap \Pi''$ is not a tangent of K and that $\delta' < \delta$. Then there is a tangent n' of K at P' not contained in $\rho \cup \Pi''$. The tangent n' is contained in a plane ρ' with $|\rho' \cap K| \leq 4$. If $\rho \cap \rho'$ is a tangent of K , then the $2q + \delta$ tangents of K at P' are contained in $\rho \cup \rho'$, a contradiction. So $\rho \cap \rho'$ is not a tangent; similarly $\rho' \cap \Pi''$ is not a tangent. Hence the number of tangents of K at P' is at least $3(q - 2)$, so $2q + \delta \geq 3q - 6$, hence $\delta \geq q - 6$, a contradiction. We conclude that $\delta' = \delta$ and that all tangents of K at P' are contained in $\rho \cup \Pi''$.

Hence, given any point $Q \in K$ there are two planes α_1 and α_2 containing all tangents of K at Q ; also $|\alpha_1 \cap K| \leq 4$ and $|\alpha_2 \cap K| \leq 4$. These planes are uniquely defined by Q , and so is $\alpha_1 \cap \alpha_2$. By Section (A) the line $\alpha_1 \cap \alpha_2$ is a tangent of K at Q . Let $\tilde{\Pi}$ be any plane containing Q , with $\tilde{\Pi} \notin \{\alpha_1, \alpha_2\}$. Then $\tilde{\Pi} \cap K$ contains at most two tangents at Q , so $|\tilde{\Pi} \cap K| \geq q$. It follows that K contains no $(q-2)$ -arcs and no $(q-1)$ -arcs.

Notice that $|\alpha_1 \cap K| + |\alpha_2 \cap K| + \delta = 3$ and remind that $-1 \leq \delta \leq 1$.

Let $\tilde{\Pi}$ be a plane containing Q , with $\tilde{\Pi} \notin \{\alpha_1, \alpha_2\}$. The arc $\tilde{\Pi} \cap K$ contains always at least one tangent of K at Q , except when $\delta = -1, k = q^2 - q + 3, |\alpha_1 \cap K| = |\alpha_2 \cap K| = 2$. So if $k \in \{q^2 - q + 1, q^2 - q + 2\}$ and if $k = q^2 - q + 3$ with $|\alpha_1 \cap K| = |\alpha_2 \cap K| + 2 = 3$ or $|\alpha_2 \cap K| = |\alpha_1 \cap K| + 2 = 3$, then $\tilde{\Pi} \cap K$ is not a $(q+2)$ -arc. If $|\alpha_1 \cap K| = |\alpha_2 \cap K| = 2, k = q^2 - q + 3$, then there is exactly one plane $\tilde{\Pi}$ containing Q for which $\tilde{\Pi} \cap K$ is a $(q+2)$ -arc.

(C.3) $k = q^2 - q + 1$

Then $\delta = 1$ and $|\Pi_1 \cap K| = |\Pi_2 \cap K| = 1$. Let $U_1, U_2 \in K$, with $U_1 \neq U_2$, and let ξ_1, ξ_2 be the planes containing U_1 intersecting K in at most 4 points. If $U_2 \in \xi_1 \cup \xi_2$, then $\delta \leq 0$, a contradiction. Hence $U_2 \notin \xi_1 \cup \xi_2$. Consequently any plane containing the line $U_1 U_2$ has more than 4 points in common with K .

Now we count the points of K in planes containing the line $U_1 U_2$. Let θ_1 be the number of planes containing $U_1 U_2$ intersecting K in a q -arc, and let θ_2 be the number of planes containing $U_1 U_2$ intersecting K in a $(q+1)$ -arc. Then

$$(59) \quad \theta_1(q-2) + \theta_2(q-1) + 2 = q^2 - q + 1, \text{ with } \theta_1 + \theta_2 = q + 1.$$

So

$$(60) \quad \theta_1(q-2) + (q+1-\theta_1)(q-1) + 2 = q^2 - q + 1,$$

that is $\theta_1 = q$ and $\theta_2 = 1$.

Now we count the number of $(q+1)$ -arcs on K , and obtain

$$(61) \quad \frac{(q^2 - q + 1)(q^2 - q)}{(q+1)q}.$$

So $q+1 \mid (q^2 - q + 1)(q-1)$, so $q+1 \mid 6$, a contradiction.

(C.4) $k = q^2 - q + 2$

Then $\delta = 0$ and $\{|\Pi_1 \cap K|, |\Pi_2 \cap K|\} = \{1, 2\}$. Let Q be any point of K and let l_Q be the tangent of K which is the intersection of the two planes α_1 and α_2 containing the $2q$ tangents of K at Q . Let $(\alpha_1 \cup \alpha_2) \cap K = \{Q, Q'\}$. Starting with Q' and $l_{Q'}$, we find the same pair $\{Q', Q\}$. It follows that K is partitioned into pairs of type $\{Q, Q'\}$. Let \mathcal{L} be the set of these $(q^2 - q + 2)/2$ pairs.

Let $\{Q, Q'\} \in \mathcal{L}$, let α_1 and α_2 be the planes containing the $2q$ tangents of K at Q , and assume that $Q' \in \alpha_1$. Then $\alpha_1 = l_Q l_{Q'}$. Let Π be a plane containing QQ' , distinct from α_1 . As Π contains a tangent of K at Q , we have $|\Pi \cap K| \leq q+1$. Counting the points of K in the planes containing QQ' , we obtain $|\Pi \cap K| = q+1$. By an easy counting we see that the planes containing l_Q , but distinct from α_1 and α_2 , intersect K in $(q+1)$ -arcs. This way there arise $q-1$ $(q+1)$ -arcs K_1, K_2, \dots, K_{q-1} , having kernels N_1, N_2, \dots, N_{q-1} on $l_Q \setminus \{Q\}$. Assume, by way of contradiction, that

$N_i = N_j, i \neq j$ and $i, j \in \{1, 2, \dots, q-1\}$. Then N_i is on at least $2q+1$ tangents of K , hence $k \leq q^2 - q + 1$, a contradiction. Let $l_Q \setminus \{N_1, N_2, \dots, N_{q-1}\} = N_Q$.

Assume, by way of contradiction, that $l_Q \cap l_{Q'} \neq N_Q$. Let $l_Q \cap l_{Q'} = N_i, i \in \{1, 2, \dots, q-1\}$, and let $R \in K_i \setminus \{Q\}$. Then $l_{Q'}R \cap K$ is a $(q+1)$ -arc with kernel N_i . Hence N_i is on at least q^2+2 tangents, a contradiction. Consequently $l_Q \cap l_{Q'} = N_Q$; similarly, $l_Q \cap l_{Q'} = N_{Q'}$.

Assume, by way of contradiction, that $l_Q \cap l_S \neq \emptyset$, with $Q \neq S$ and $\{Q, S\} \notin \mathcal{L}$. Let $\{Q, Q'\}$ and $\{S, S'\}$ be elements of \mathcal{L} . Now we count the number of tangents of K containing $l_Q \cap l_S = M$. The arc $l_Q l_S \cap K$ is a $(q+1)$ -arc with kernel M , so $l_Q l_S$ contains $q+1$ tangents of K through M ; the arc $l_Q S' \cap K$ is a $(q+1)$ -arc, and as the line MS' of the plane $l_S S'$ is a tangent of K , the point M is the kernel of $l_Q S' \cap K$, so $l_Q S'$ contains $q+1$ tangents of K through M ; similarly the plane $l_S Q'$ contains $q+1$ tangents of K through M . Hence M is contained in more than $2q$ tangents of K , clearly a contradiction. It follows that if $l_Q \cap l_S \neq \emptyset$, with $Q \neq S$, then $\{Q, S\} \in \mathcal{L}$.

Let $\{Q, S\} \notin \mathcal{L}$, with Q and S distinct points of K . Then $l_Q \cap l_S = \emptyset$. Now we count the points of K in the planes containing the line QS . Let θ_1 be the number of planes which contain QS and intersect K in a q -arc, and let θ_2 be the number of planes which contain QS and intersect K in a $(q+1)$ -arc. Hence

$$(62) \quad \theta_1(q-2) + \theta_2(q-1) + 2 = q^2 - q + 2, \text{ with } \theta_1 + \theta_2 = q+1.$$

So $\theta_1 = q-1$ and $\theta_2 = 2$. The 2 planes containing QS and intersecting K in a $(q+1)$ -arc are the planes $l_Q S$ and $l_S Q$.

Let $\{Q, S\} \in \mathcal{L}$ and let $l_Q \cap l_S = N$. Then N is kernel of no one of the $q-1$ $(q+1)$ -arcs defined by planes containing the tangent l_Q and of no one of the $q-1$ $(q+1)$ -arcs defined by planes containing the tangent l_S . So for any line $n \notin \{l_Q, l_S\}$ containing N we have $|n \cap K| \in \{0, 2\}$. Let $n \cap K = \{U, U'\}$.

First, assume that $\{U, U'\} \notin \mathcal{L}$. Then $|l_U U' \cap K| = |l_{U'} U \cap K| = q+1$. As $|l_Q U \cap K| = |l_S U \cap K| = q+1$, the planes $l_Q U$ and $l_S U$ are the two planes containing UU' and intersecting K in a $(q+1)$ -arc. Hence $\{l_Q U, l_S U\} = \{l_U U', l_{U'} U\}$. So we may assume that $l_Q U = l_U U'$ and $l_S U = l_{U'} U$. Consequently $l_Q \cap l_U \neq \emptyset$ and $l_S \cap l_{U'} \neq \emptyset$, that is, $\{Q, U\} \in \mathcal{L}$ and $\{S, U'\} \in \mathcal{L}$. Hence $|l_Q l_U \cap K| = |l_S l_{U'} \cap K| = 2$, clearly a contradiction as $Q, U, U' \in l_Q l_U$.

It follows that $\{U, U'\} \in \mathcal{L}$. So for any pair $\{T, T'\} \in \mathcal{L}$, with $\{T, T'\} \neq \{Q, S\}$, we have $N \in TT'$. Let n', n'' be distinct lines containing N with $n' \neq n \neq n''$ and $n', n'' \notin \{l_Q, l_S\}$. Assume also that $n' \cap K = \{V, V'\}$ and $n'' \cap K = \{W, W'\}$. Then $\{V, V'\} \in \mathcal{L}$ and $\{W, W'\} \in \mathcal{L}$. By the foregoing the lines VV', WW', QS contain N , clearly a contradiction.

$$(C.5) \quad k = q^2 - q + 3$$

Let P be any point of K and let l_P be the tangent of K which is the intersection of the two planes Π_1, Π_2 containing the $2q-1$ tangents of K at P . Two cases are considered.

$$(C.5.1) \quad \Pi_1 \cap K = \{P, P', P''\}, \Pi_2 \cap K = \{P\}$$

Then K contains no plane $(q+2)$ -arcs containing P . Let l be a tangent of K at P , with $l \in \Pi_1$ and $l \neq l_P$. We count the points of K in planes containing l . Let θ_1 be

the number of planes containing l and intersecting K in a $(q+1)$ -arc, and let θ_2 be the number of planes containing l and intersecting K in a q -arc. Then

$$(63) \quad \theta_1 q + \theta_2(q-1) + 3 = q^2 - q + 3, \text{ with } \theta_1 + \theta_2 = q.$$

Hence $\theta_1 = 0$ and $\theta_2 = q$. Let $\tilde{\Pi}_1, \tilde{\Pi}_2, \dots, \tilde{\Pi}_q$ be the planes containing l and intersecting K in a q -arc, let $\tilde{\Pi}_i \cap K = K_i$, let C_i be the $(q+2)$ -arc extending K_i and let $C_i \cap l = \{P, N_i\}$, with $i = 1, 2, \dots, q$. Assume that for some $i \in \{1, 2, \dots, q\}$ we have $N_i \notin P'P''$. The number of tangents of K containing N_i is at least

$$(64) \quad q + (q-1) + 2 = 2q + 1,$$

a contradiction. Hence $N_1 = N_2 = \dots = N_q = l \cap P'P''$. Then the number of tangents of K containing N_1 is at least

$$(65) \quad q(q-1) + 1 = q^2 - q + 1,$$

again a contradiction.

$$(C.5.2) \quad \Pi_1 \cap K = \{P, P'\}, \Pi_2 \cap K = \{P, P''\}$$

By (C.5.1), for each point $Q \in K$ the two planes α_1, α_2 through Q intersecting K in at most four points, intersect K in exactly two points. If $\alpha_1 \cap K = \{Q, Q'\}$ and $\alpha_2 \cap K = \{Q, Q''\}$, then the plane $QQ'Q''$ is the only plane on Q intersecting K in a $(q+2)$ -arc. Hence the $(q+2)$ -arcs on K partition K . So

$$(66) \quad q + 2|q^2 - q + 3, \text{ so } q + 2|q - 7, \text{ so } q + 2|9,$$

a contradiction.

Now the theorem is proved. ■

4. COROLLARIES

We are grateful to T. Szőnyi for bringing reference [12] to our attention which, in combination with Theorem 1.6, gives the following considerable improvement of the bound in Theorem 1.6; see also Remark 4.4.

Theorem 4.1.

$$(67) \quad m'_2(3, q) < q^2 - 2q + 3\sqrt{q} + 2, q \text{ even}, q \geq 2048.$$

Proof. In [12] it is proved that there does not exist a complete k -cap in $\text{PG}(3, q)$, q even, $q \geq 64$, for which

$$(68) \quad k \in [q^2 - (a-1)q + a\sqrt{q} + 2 - a + \frac{a(a-1)}{2}, q^2 - (a-2)q - a^2\sqrt{q}]$$

where a is an integer which satisfies

$$(69) \quad 2 \leq a \leq \frac{-2\sqrt{q} + 3 + \sqrt{16q\sqrt{q} + 12q - 44\sqrt{q} - 7}}{4\sqrt{q} + 2}.$$

Putting $a = 3$, the desired result immediately follows from Theorem 1.6. ■

Theorem 4.2. (i) $m_2(4, 4) = 41$,

$$(ii) \quad m_2(4, 8) \leq 479,$$

$$(iii) \quad m_2(4, q) < q^3 - q^2 + 2\sqrt{5}q - 8, q \text{ even}, q > 8.$$

Proof. For $q = 4$, see [5]. Assume, by way of contradiction, that K is a k -cap of $\mathbf{PG}(4, 8)$ with $k > 479$, or a k -cap of $\mathbf{PG}(4, q)$, q even and $q > 8$, with $k > q^3 - q^2 + 2\sqrt{5}q - 8$. At each of its points the cap K has $t = q^3 + q^2 + q + 2 - k$ tangents. Hence we assume that $t < 107$ for $q = 8$ and $t < 2q^2 + (1 - 2\sqrt{5})q + 10$ for $q > 8$. We obtain a contradiction in several stages.

I K contains no plane q -arc

Similar to the reasoning in Section I in the proof of Theorem 6.27 in [9].

II There exists no solid δ such that $q^2 + 1 > |\delta \cap K| > q^2 + (1 - \sqrt{5})q + 5$

Suppose δ exists. Let $\delta \cap K = K'$. Then K' can be completed to an ovoid O of δ , by Theorem 3.1. Let $N \in O \setminus K'$ and let $N' \in K'$. Consider the $q + 1$ planes of δ through NN' . Since each of these planes meets O in a $(q + 1)$ -arc, each plane meets K' in at most a q -arc. By I, there is no q -arc on K ; so each plane meets K' in at most a $(q - 1)$ -arc.

Assume, by way of contradiction, that none of these intersections is a $(q - 1)$ -arc. Therefore a count of the points on K' gives

$$(70) \quad |K'| \leq (q + 1)(q - 3) + 1,$$

whence

$$(71) \quad q^2 + (1 - \sqrt{5})q + 5 < q^2 - 2q - 2,$$

so

$$(72) \quad (3 - \sqrt{5})q + 7 < 0,$$

a contradiction.

So we may assume that for one of the planes δ through NN' , say Π , we have $|\Pi \cap K'| = q - 1$. Now we consider all solids of $\mathbf{PG}(4, q)$ containing the plane Π . Let θ be the number of solids Π' for which $|\Pi' \cap K| > q^2 + (1 - \sqrt{5})q + 5$, so $q + 1 - \theta$ is the number of solids Π'' for which $|\Pi'' \cap K| < q^2 + (1 - \sqrt{5})q + 5$. We have $\theta \geq 1$.

First, assume $\theta \geq 2$. So there are at least two solids Π'_1, Π'_2 containing Π such that $|\Pi'_i \cap K| > q^2 + (1 - \sqrt{5})q + 5$, with $i = 1, 2$. By Theorem 3.1 $\Pi'_i \cap K$ can be completed to an ovoid O_i of Π'_i , $i = 1, 2$. So $O_i \cap \Pi$ is a $(q + 1)$ -arc $(\Pi \cap K') \cup \{N'_i, N''_i\}$, $i = 1, 2$. Since $\Pi \cap K'$ can be contained in no more than three $(q + 1)$ -arcs, contained in a common $(q + 2)$ -arc, we have $|\{N'_1, N''_1\} \cap \{N'_2, N''_2\}| \geq 1$. Assume $N'_1 = N'_2$. So the number of tangents of K containing N'_1 is at least

$$(73) \quad 2(q^2 + (1 - \sqrt{5})q + 5 - q + 1) + q - 1,$$

so

$$(74) \quad 2q^2 + (1 - 2\sqrt{5})q + 11,$$

a contradiction.

Finally, assume that $\theta = 1$. Counting the points of K in the $q + 1$ solids, we obtain

$$(75) \quad k < q(q^2 + (1 - \sqrt{5})q + 5 - q + 1) + (q^2 - 1),$$

that is,

$$(76) \quad k < q^3 + (1 - \sqrt{5})q^2 + 6q - 1.$$

Hence, for $q > 8$,

$$(77) \quad q^3 - q^2 + 2\sqrt{5}q - 8 < q^3 + (1 - \sqrt{5})q^2 + 6q - 1,$$

so

$$(78) \quad 0 < (2 - \sqrt{5})q^2 + (6 - 2\sqrt{5})q + 7,$$

a contradiction. For $q = 8$, there arises $479 < 479$, a contradiction.

III For a point N not in K , there do not exist planes Π_1 and Π_2 such that $\Pi_1 \cap \Pi_2 = N$ and such that $\Pi_i \cap K$ is a $(q+1)$ -arc with nucleus N for $i = 1, 2$
Similar to the reasoning in Section III in the proof of Theorem 6.27 in [9].

IV The tangents through any point Q off K lie in a solid

Similar to the reasoning in Section IV in the proof of Theorem 6.27 in [9].

V The final contradiction is obtained by counting the tangents of K

Similar to the reasoning in Section V in the proof of Theorem 6.27 in [9]. ■

Theorem 4.3. For q even, $q > 2$, $n \geq 5$,

- (i) $m_2(n, 4) \leq \frac{118}{3} \cdot 4^{n-4} + \frac{5}{3}$
- (ii) $m_2(n, 8) \leq 478 \cdot 8^{n-4} - 2(8^{n-5} + \dots + 8 + 1) + 1,$
- (iii) $m_2(n, q) < q^{n-1} - q^{n-2} + 2\sqrt{5}q^{n-3} - 9q^{n-4} - 2(q^{n-5} + \dots + q + 1) + 1$, for $q > 8$.

Proof This follows directly from Theorem 1.1, Theorem 4.2 and Theorem 6.14(ii) in [9]. ■

Remark 4.4. The bound in Theorem 4.1 leads to considerable improvements of Theorem 4.2 and Theorem 4.3. We just mention these bounds, but the proofs are the theme of a subsequent paper.

For q even, $q \geq 2048$,

$$(79) \quad m_2(4, q) < q^3 - 2q^2 + 3q\sqrt{q} + 8q - 9\sqrt{q} - 6.$$

For q even, $q \geq 2048$, $n \geq 5$,

$$(80) \quad m_2(n, q) < q^{n-1} - 2q^{n-2} + 3q^{n-3}\sqrt{q} + 8q^{n-3} - 9q^{n-4}\sqrt{q} - 7q^{n-4} - 2(q^{n-5} + \dots + q + 1) + 1.$$

5. REMARK

The bound in the MAIN THEOREM is better than the bound of Chao, see [3]. In 2014 Cao and Ou, see [2], published the bound $k < q^2 - 2q + 8$ (q even and $q \geq 128$), which is better than ours. I did not follow some reasoning in their proof, so I sent two mails to one of the authors explaining why I think Section 1.3 of the proof is not correct. Unfortunately I never received an answer.

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